Geometrizing Galois Theory: Deep Ramification and Vector Bundles.

Recent Developments in Arithmetic and Algebraic Geometry in Positive Characteristic.

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Introduction

Galois Correspondence in $\mathbb{C}_p$

Let K be a finite extension of $\mathbb{Q}_p$.

Theorem 1 (Ax-Sen-Tate)

If L/K is an algebraic extension then $\widehat{L} = \mathbb{C}_p^{G_L}$.

We have a antitone map betwen the following POSETs

$$\mathrm{Sub}_{cl}(G_K) \longrightarrow \mathrm{Sub}(\mathbb{C}_p),\ H \mapsto \mathbb{C}_p^H$$

Theorem 2 (Iovita-Zaharescu)

There are a bijection between the set of closed subgroups of G_K and the set of complete valued sub-fields of $\mathbb{C}_p$.

The De-Rham Period Ring

Let $R = \varprojlim_{\varphi} \frac{\mathcal{O}_{\mathbb{C}_K}}{p\mathcal{O}_{\mathbb{C}_K}}$, define

- $\mathbb{A}_{inf} := W(R)$. This ring is eqquiped with a continuous morphisms

$$\theta \colon \mathbb{A}_{inf} \longrightarrow \mathcal{O}_{\mathbb{C}_K}$$

  whose kernel is a principal ideal.

- $$\mathbb{B}^{+}_{dR} := \left(\varprojlim_{n} \frac{\mathbb{A}_{inf}[1/p]}{\ker(\theta)^n}, \widehat{\ker(\theta)} \right)$$

Galois Correspondence in $\mathbb{B}_{dR}^{+}$

$\overline{K}$ with the p-adic topology satisfies $\widehat{\overline{K}} \cong \mathbb{C}_p$.

There a sequence of descendent subrings $\mathcal{O}^{(n)} \subseteq \mathcal{O}_K$ such that $\overline{K}$ with the topology defined by the fundamental system $\{p^m \mathcal{O}^{(n)}\}_{n,m \in \mathbb{N}}$ satisfices:

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$$\begin{array}{ccc} \overline{K} & \hookrightarrow & \mathbb{B}_{dR}^{+} \\ & \searrow & \downarrow \\ & & \mathbb{C}_p \end{array} \quad , \quad \mathbb{B}_{dR}^{+}/\xi\mathbb{B}_{dR}^{+} \cong \mathbb{C}_p$$

$$L \hookrightarrow (\mathbb{B}_{dR}^{+})^{G_L}$$

Definition 3

We say that L satisfies the Galois Correspondence of $\mathbb{B}_{dR}^{+}$ if L is dense in $(\mathbb{B}_{dR}^{+})^{G_L}$.

The Density Problem.

Non Deeply ramified extensions

Definition 4 (Deeply Ramified Extension)

Let L/K be an algebraic extension. We say that L is a Deeply Ramified Extension if the field $\widehat{L}$ is perfectoid.

Theorem 5 (Iovita-Zaharescu)

If L is not deeply ramified extension then L satisfies the Galois Correspondence of $\mathbb{B}_{dR}^{+}$.

Denote $B_n := \mathbb{B}_{dR}^{+}/\xi^n\mathbb{B}_{dR}^{+}$, then

$$\mathbb{B}_{dR}^{+} \cong \varprojlim_{n} B_n.$$

Deeply ramified extensions

Definition 6 (Property (*) at level n)

Let L be a deeply ramified extension of K and $n \in \mathbb{N}$ with $n > 2$. We say that L has **property (*)** *at level n if $T_p(d_{n-1}(\mathcal{O}_L^{(n-2)})) \neq 0$. Equivalently, there exists a sequence $\{\alpha_r\}_r$ with $\alpha_r \in \mathcal{O}_L^{(n-2)}$ such that $0 \neq d_{n-1}\alpha_r = pd_{n-1}\alpha_{r+1}$ for all r, which implies the existence of a non-zero element in $J_n \cap \wedge^n L$.*

Theorem 7 (Theorem 7.1, Iovita-Zaharescu)

Let L be an algebraic extension of K and $n \in \mathbb{N}$, $n > 2$[cite: 1]. If L is deeply ramified, then the following are equivalent[cite: 1]:

(a) $(B_m)^{G_L} = \wedge^m L$ *for all $m \leq n$ (i.e., L satisfies the Galois correspondence at all levels $m \leq n$).*

(b) *L has property (*) at all levels $m \leq n$.*

Bloch-Kato subgroups.

The Bloch-Kato Subgroups

Let V be a p-adic representation of G_K and K' a finite extension of K.

Using Fontaine's period rings, we classify cohomology classes with good geometric properties:

- **Exponential:**
$$H^1_e(K', V) = \ker\left(H^1(K', V) \to H^1(K', B_e \otimes_{\mathbb{Q}_p} V)\right)$$
- **Finite (Crystalline):**
$$H^1_f(K', V) = \ker\left(H^1(K', V) \to H^1(K', B_{cris} \otimes_{\mathbb{Q}_p} V)\right)$$
- **Geometric (De Rham):**
$$H^1_g(K', V) = \ker\left(H^1(K', V) \to H^1(K', B_{dR} \otimes_{\mathbb{Q}_p} V)\right)$$

Infinite Level (Iwasawa Theory):
For a $\mathbb{Z}_p$-lattice $T \subset V$ and an infinite extension L/K, we take the direct limit over finite subextensions $K' \subset L$:
$$H^1_*(L, V/T) = \varinjlim_{K'} H^1_*(K', V/T)$$

The Fargues-Fontaine Equivalence

- An elliptic curve E is given by the Weierstrass equation:

$$y^2 = x^3 + Ax + B$$

- It naturally yields a p-adic Galois representation via its Tate module:

$$V_p(E) = T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

The Bridge: Equivalence of Categories
How do we study $V_p(E)$ using the Fargues-Fontaine curve X_{FF}?
There is a fundamental equivalence of categories:

$$\mathbf{Rep}_{\mathbb{Q}_p}(G_K) \xrightarrow{\sim} \mathbf{Bun}^0_{FF}(G_K)$$

- **The Functor:** $V \longmapsto \mathcal{E}(V) = \mathcal{O}_{X_{FF}} \otimes_{\mathbb{Q}_p} V$
- **Meaning:** Any p-adic representation V (like $V_p(E)$) corresponds to a G_K-equivariant, semi-stable vector bundle of slope 0 over X_{FF}.

Ponsinet's Theorem & The Density Problem

Ponsinet's Theorem (2020):
Let V be a de Rham representation with Hodge-Tate weights ≤ 1. If $\widehat{L}$ is a perfectoid field, the exponential Bloch-Kato subgroup is exactly:

$$H^1_e(L, V/T) = \mathrm{Im}\left(H^1(L, V_0/T_0) \xrightarrow{\lambda_L} H^1(L, V/T)\right)$$

(Proved by classifying the vector bundle $\mathcal{E}^+(V)$ over the Fargues-Fontaine curve, bypassing the topological pathologies of B_{dR}).

When does Density hold? (Examples of Fields):

- **Iovita-Zaharescu:** Density in B^+_{dR} holds for extensions L/K that are **not deeply ramified** (e.g., any finite extension of K).
- **Ponsinet / Perfectoid Fields:** For deeply ramified extensions (like the cyclotomic extension $L = \bigcup K(\zeta_{p^n})$), density in B^+_{dR} **fails**. However, density at level 1 (in $\mathbb{C}_p$) still holds by the Ax-Sen-Tate theorem.

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